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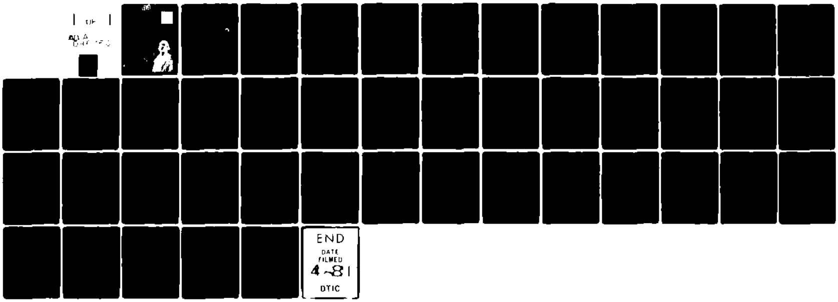

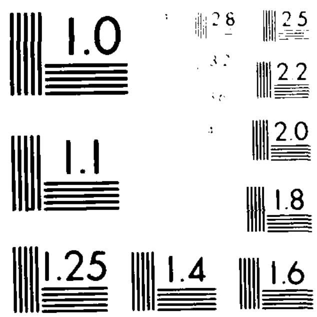

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OPTIMAL VALUE DIFFERENTIAL STABILITY BOUNDS UNDER THE MANGASARIAN-FROMOVITZ CONSTRAINT QUALIFICATION

by

Anthony V. Fiacco

Serial T-435
1 October 1980

The George Washington University
School of Engineering and Applied Science
Institute for Management Science and Engineering

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20 Abstract (continued)

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1. Introduction

The first- and second-order variation of the optimal value of a general nonlinear program under quite arbitrary parametric perturbations has been investigated by Hogan [15], Armacost and Fiacco [1,2,3], Fiacco [9], and Fiacco and McCormick [11]. In [2] the optimal value function is shown, under strong conditions, to be twice continuously differentiable, with respect to the problem parameters, with its parameter gradient (Hessian) equal to the gradient (Hessian) of the Lagrangian of the problem. Armacost and Fiacco [1] have also obtained first- and second-order expressions for changes in the optimal value function as a function of right-hand side perturbations.

A number of results relating to the differential stability of the optimal value function have also been obtained, generally associated with the existence of directional derivatives or bounds on the directional derivative limit quotient. Danskin [6,7] provided one of the earliest characterizations of the differential stability of the optimal value function of a mathematical program. Addressing the problem minimize $f(x,\varepsilon)$ subject to $x \in S$, S some topological space, ε in E^k, Danskin derived conditions under which the directional derivative of f* exists and also determined its representation. This result has wide applicability in the sense that the constraint space, S, can be any compact topological space. However, the result is restricted to a constraint set that does not vary with the parameter ε. For the special case in which S is defined by inequalities involving a parameter, $g_i(x,\varepsilon) \geq 0$ for $i = 1,\ldots,m$, where f is convex and the g_i are concave on S, Hogan [15] has given conditions that imply that the directional derivative of f* exists and is finite in all directions.

For programs without equality constraints, Rockafellar [19] has shown that, under certain second-order conditions, the optimal value function satisfies a stability of degree two. Under this stability property, bounds on the directional derivative of f* can be derived. For convex programming problems, Gol'stein [14] has shown that a saddle point condition is satisfied by the directional derivative of f*. Gauvin and Tolle [13], not assuming convexity, but limiting their analysis to right-hand side perturbations, extended the work of Gol'stein and provide sharp upper and lower bounds on the directional derivative limit quotient of f*, assuming the Mangasarian-Fromovitz constraint qualification and without requiring the existence of second-order conditions. Optimal value sensitivity results for infinite dimensional programs have recently been obtained by Maurer [17,18].

The purpose of this paper is to refine and continue the preliminary but incomplete study conducted by Fiacco and Kutzler [10] that extends the results of Gauvin and Tolle [13] to the general inequality constrained mathematical program in which a parameter appears arbitrarily in the constraints and the objective function. We complete this extension and obtain, for the general inequality-equality problem, the Gauvin-Tolle upper and lower bounds on the directional derivative limit quotient of the optimal value function. In a paper essentially simultaneous to [10], Gauvin and Dubeau [12] have independently obtained the differential bounds under the same assumptions invoked here but using a different method of proof.

Sections 1 through 3, through Theorem 3.7, and Sections 5 and 6 are taken more or less intact from [10] and are reported here for completeness. Theorem 4.1 and Corollary 4.2 were also obtained in [10]. The remaining results, Lemmas 3.8, 3.10, 4.3, 4.4 and Theorems 3.9, 3.11, 4.6, 4.8, and 4.9, extend the reduction approach introduced in [10] and complete the development of the theory for the general inequality-equality constrained parametric problem.

2. Notation and Definitions

In this paper we shall be concerned with mathematical programs of the form:

$$\min_x f(x,\varepsilon) \qquad\qquad P(\varepsilon)$$
$$\text{s.t. } g_i(x,\varepsilon) \geq 0 \ (i=1,\ldots,m),\ h_j(x,\varepsilon) = 0 \ (j=1,\ldots,p),$$

where $x \in E^n$ is the vector of decision variables, ε is a parameter vector in E^k, and the functions f, g_i and h_j are once continuously differentiable on $E^n \times E^k$. The feasible region of problem $P(\varepsilon)$ will be denoted $R(\varepsilon)$ and the set of solutions $S(\varepsilon)$. The m-vector whose components are $g_i(x,\varepsilon)$, $i = 1,\ldots,m$, and the p-vector whose components are $h_j(x,\varepsilon)$, $j = 1,\ldots,p$, will be denoted by $g(x,\varepsilon)$ and $h(x,\varepsilon)$, respectively.

Following usual conventions the gradient, with respect to x, of a once differentiable real-valued function $f:E^n x E^k \to E^1$ is denoted $\nabla_x f(x,\varepsilon)$ and is taken to be the row vector $[\partial f(x,\varepsilon)/\partial x_1,\ldots,\partial f(x,\varepsilon)/\partial x_n]$.

If $g(x,\varepsilon)$ is a vector-valued function, $g:E^n x E^k \to E^m$, whose components $g_i(x,\varepsilon)$ are differentiable in x, then $\nabla_x g(x,\varepsilon)$ denotes the m x n Jacobian matrix of g whose ith row is given by $\nabla_x g_i(x,\varepsilon)$, $i = 1,\ldots,m$. The transpose of the Jacobian $\nabla_x g(x,\varepsilon)$ will be denoted $\nabla_x' g(x,\varepsilon)$. Differentiation with respect to the vector ε is denoted in a completely analogous fashion. Transposition of vectors and matrices is denoted by a prime.

The Lagrangian for $P(\varepsilon)$ will be written

$$L(x,\mu,\omega,\varepsilon) = f(x,\varepsilon) - \sum_{i=1}^{m} \mu_i g_i(x,\varepsilon) + \sum_{j=1}^{p} \omega_j h_j(x,\varepsilon),$$

and the set of Kuhn-Tucker vectors corresponding to the decision vector x will be given by

$$K(x,\varepsilon) = \{(\mu,\omega)\varepsilon E^m x E^p : \nabla_x L(x,\mu,\omega,\varepsilon) = 0,\ \mu_i \geq 0,\ \mu_i g_i(x,\varepsilon) = 0,\ i=1,\ldots,m\}.$$

Writing a solution vector as a function of the parameter ε, the index set for inequality constraints which are binding at a solution $x(\varepsilon)$ is denoted by $B(\varepsilon) = \{i:g_i(x(\varepsilon),\varepsilon) = 0\}$. Finally, the optimal value function will be defined as

$$f^*(\varepsilon) = \min \{f(x,\varepsilon):x\varepsilon R(\varepsilon)\}.$$

Throughout this paper we shall make use of the well known Mangasarian-Fromovitz Constraint Qualification (MFCQ) which holds at a point $x\varepsilon R(\varepsilon)$ if:

i) there exists a vector $\tilde{y}\varepsilon E^n$ such that

$$\nabla_x g_i(x,\varepsilon)\, \tilde{y} > 0 \text{ for } i \text{ such that } g_i(x,\varepsilon) = 0 \text{ and} \quad (2.1)$$

$$\nabla_x h_j(x,\varepsilon)\, \tilde{y} = 0 \text{ for } j=1,\ldots,p; \text{ and} \quad (2.2)$$

ii) the gradients $\nabla_x h_j(x,\varepsilon)$, $j=1,\ldots,p$, are linearly independent.

We will have occasion to make use of various continuity properties for both real-valued functions and point-to-set maps. There are several related definitions of the indicated properties. The ones most suited to our purpose follow. The reader interested in more detail is referred to Berge [5] and Hogan [16].

Definition 2.1. Let ϕ be a real-valued function defined on the space X.

i) ϕ is said to be lower semicontinuous (lsc) at a point $x_o \varepsilon X$ if $\underline{\lim}_{x \to x_o} \phi(x) \geq \phi(x_o)$.

ii) ϕ is said to be upper semicontinuous (usc) at a point $x_o \varepsilon X$ if $\overline{\lim}_{x \to x_o} \phi(x) \leq \phi(x_o)$.

Using these definitions, one readily sees that a real-valued function ϕ is continuous at a point if and only if it is both upper and lower semi-continuous at that point.

Definition 2.2. Let $\phi: X \to Y$ be a point-to-set mapping and let $\{\varepsilon_n\} \subset X$, with $\varepsilon_n \to \bar{\varepsilon}$ in X as $n \to \infty$.

i) ϕ is said to be open at a point $\bar{\varepsilon}$ of X if, for each $\bar{x} \varepsilon \phi(\bar{\varepsilon})$, there exists a value n_o and a sequence $\{x_n\} \subset Y$ with $x_n \varepsilon \phi(\varepsilon_n)$ for $n \geq n_o$ and $x_n \to \bar{x}$.

ii) ϕ is said to be closed at a point $\bar{\varepsilon}$ of X if $x_n \varepsilon \phi(\varepsilon_n)$ and $x_n \to \bar{x}$ together imply that $\bar{x} \varepsilon \phi(\bar{\varepsilon})$.

Definition 2.3. A point-to-set mapping $\phi: X \to Y$ is said to be uniformly compact near a point $\bar{\varepsilon}$ of X if the closure of the set $\bigcup_{\varepsilon \text{ in } N(\bar{\varepsilon})} \phi(\varepsilon)$ is compact for some neighborhood $N(\bar{\varepsilon})$ of $\bar{\varepsilon}$.

In Section 3 we apply a reduction of variables technique to $P(\varepsilon)$ which transforms that program to a locally equivalent program involving only inequality constraints. This approach simplifies the derivation of intermediate results which are needed to derive the bounds on the directional derivative limit quotients of $f^*(\varepsilon)$ given in Section 4. A demonstration of the results is provided in the example of Section 5. Section 6 concludes with a few remarks concerning related results.

3. Reduction of Variables

In $P(\varepsilon)$, if the rank of the Jacobian, $\nabla_x h$, with respect to x of the (first n) equality constraints in a neighborhood of a solution is equal to n, then the given solution is completely determined as a solution of the system of equations $h_j(x,\varepsilon) = 0$, $j = 1,\ldots,n$, and the (locally unique) solution, $x(\varepsilon)$, of this system near $\varepsilon = 0$ is then completely characterized by the usual implicit function theorem. We are here interested in the less structured situation and hence assume that the rank of $\nabla_x h$ is less than n. Since we shall be making use of MFCQ, this entails the assumption that the number p of equality constraints is less than n. If there are no equality constraints in a particular formulation of $P(\varepsilon)$, simply suppress reference to h in the following development. Otherwise, we take advantage of the linear independence assumption to eliminate the equalities, again using an implicit function theorem.

Let $x = (x_D, x_I)$, where $x_D \varepsilon E^p$ and $x_I \varepsilon E^{n-p}$. Reordering variables if necessary, if $h(x^*,\varepsilon^*) = 0$ and MFCQ holds at x^*, then we may assume that $\nabla_{x_D} h$ is nonsingular at (x^*,ε^*). Then the usual implicit function theorem results hold: there exists an open set $N^* \subset E^{n-p} \times E^k$ containing (x_I^*,ε^*) such that the system of equations $h(x_D,x_I,\varepsilon) = 0$ can be solved for x_D in terms of x_I and ε for any (x_I,ε) in N^*. Furthermore, this representation is unique, the resulting function $x_D = x_D(x_I,\varepsilon)$ is continuous, and $x_D^* = x_D(x_I^*,\varepsilon^*)$. Thus, in N^*, the system $h(x_D,x_I,\varepsilon) = 0$ is satisfied identically by the function $x_D = x_D(x_I,\varepsilon)$. Under our additional assumption that h is once continuously differentiable in x_I and ε, $x_D(x_I,\varepsilon)$ is also once continuously differentiable in x_I and ε.

Applying this result to $P(\varepsilon)$ at x^*, since we have $\tilde{h}(x_I,\varepsilon) \equiv h[x_D(x_I,\varepsilon),x_I,\varepsilon] \equiv 0$ in N^*, this problem can be reduced locally to one involving only inequality constraints:

$$\min_{x_I} \tilde{f}(x_I,\varepsilon)$$

$$\text{s.t.} \quad \tilde{g}_i(x_I,\varepsilon) \geq 0 \qquad (i=1,\ldots,m), \qquad\qquad \tilde{P}(\varepsilon)$$

$$\text{and} \quad (x_I,\varepsilon) \in N^*,$$

where $\tilde{f}(x_I,\varepsilon) \equiv f[x_D(x_I,\varepsilon),x_I,\varepsilon]$ and $\tilde{g}_i(x_I,\varepsilon) \equiv g_i[x_D(x_I,\varepsilon),x_I,\varepsilon]$ for $i=1,\ldots,m$, and where the minimization is now performed over the n-p dimensional vector x_I. The programs P and $\tilde{P}$ are locally equivalent, for (x,ε) in a neighborhood T^* of $(x^*,0)$ and for (x_I,ε) in N^*, in the sense that the point $x(\varepsilon) \varepsilon E^n$, with $(x(\varepsilon),\varepsilon)$ in T^* and $x(\varepsilon) = (x_D(\varepsilon),x_I(\varepsilon))$, satisfies the Karush-Kuhn-Tucker first-order necessary conditions for an optimum of $P(\varepsilon)$ if and only if the point $x_I(\varepsilon)$, with $(x_I(\varepsilon),\varepsilon) \in N^*$, satisfies those conditions for $\tilde{P}(\varepsilon)$, where $x_D(x_I,\varepsilon)$ is as given above. Furthermore, in the given neighborhoods, $x(\varepsilon) = (x_D(x_I(\varepsilon),\varepsilon),x_I(\varepsilon))$ is a local solution of $P(\varepsilon)$ if and only if $x_I(\varepsilon)$ is a local solution of $\tilde{P}(\varepsilon)$.

We first observe that the Mangasarian-Fromovitz constraint qualification for $P(\varepsilon)$ is inherited by the reduced problem $\tilde{P}(\varepsilon)$. For simplicity in notation, and without loss of generality, assume that $\varepsilon^* = 0$, and assume that the components of x have been relabeled so that $x = (x_D,x_I)$ and $\nabla_{x_D} h(x_D^*,x_I^*,0)$ is nonsingular. The next result is easily obtained by invoking MFCQ at $(x,\varepsilon) = (x^*,0)$, partitioning the MFCQ vector $\tilde{y} = (\tilde{y}_D,\tilde{y}_I)$ in conformance with $x^* = (x_D^*,x_I^*)$, differentiating $\tilde{h}$ and $\tilde{g}$ with respect to x_I, and applying the assumptions. Corresponding to the notation for $P(\varepsilon)$, we denote the feasible region, solution set and optimal value of $\tilde{P}(\varepsilon)$ by $\tilde{R}(\varepsilon)$, $\tilde{S}(\varepsilon)$ and $\tilde{f}^*(\varepsilon)$, respectively. Other corresponding problem constituents will be similarly denoted.

<u>Lemma 3.1</u>. If $g,h\varepsilon C^1$, then MFCQ holds at $x^*\varepsilon R(0)$, the feasible region of $P(0)$, with $\tilde{y} = (\tilde{y}_D,\tilde{y}_I) \varepsilon E^n$ the associated vector, where $\tilde{y}_D \varepsilon E^p$ and $\tilde{y}_I \varepsilon E^{n-p}$, if and only if MFCQ holds at the point $x_I^* \varepsilon \tilde{R}(0)$, the feasible region of $\tilde{P}(0)$, with vector $\tilde{y}_I$.

<u>Proof</u>. Suppose that MFCQ holds for $P(0)$ at $(x^*,0) = (x_D^*,x_I^*,0)$ with $\tilde{y} = (\tilde{y}_D,\tilde{y}_I)$ the associated vector. Writing $\nabla_x h$ as $\left[\nabla_{x_D} h : \nabla_{x_I} h\right]$, we see that (2.2) can be expressed as:

$$\nabla_{x_D} h(x^*,0)\ \tilde{y}_D + \nabla_{x_I} h(x^*,0)\ \tilde{y}_I = 0. \tag{3.1}$$

Since we have assumed that $\nabla_{x_D} h(x^*,0)$ is nonsingular, we can solve for $\tilde{y}_D$ in (3.1) and obtain:

$$\tilde{y}_D = -\left[\nabla_{x_D} h(x^*,0)\right]^{-1} \nabla_{x_I} h(x^*,0)\ \tilde{y}_I. \tag{3.2}$$

Now, denoting the inequality constraints of $\tilde{P}(0)$ by $\tilde{g}_i$, i.e., $\tilde{g}_i = g_i(x_D(x_I,0),x_I,0)$, $i = 1,\ldots,m$, by differentiating with respect to x_I we obtain:

$$\nabla_{x_I}\tilde{g}_i = \nabla_{x_D} g_i\ \nabla_{x_I} x_D + \nabla_{x_I} g_i,$$

or

$$\nabla_{x_I}\tilde{g}_i = \nabla_x g_i \begin{bmatrix} \nabla_{x_I} x_D \\ I \end{bmatrix}. \tag{3.3}$$

Multiplying by $\tilde{y}_I$ in (3.3) we have:

$$\nabla_{x_I}\tilde{g}_i\ \tilde{y}_I = \nabla_x g_i \begin{bmatrix} \nabla_{x_I} x_D \\ I \end{bmatrix} \tilde{y}_I. \tag{3.4}$$

But $h(x_D(x_I,0),x_I,0) \equiv 0$ so that

$$\nabla_{x_D} h(x_D(x_I,0),x_I,0)\ \nabla_{x_I} x_D + \nabla_{x_I} h(x_D(x_I,0),x_I,0) = 0,$$

and since $\nabla_{x_D} h(x_D(x_I,0),x_I,0)$ is nonsingular, we obtain:

$$\nabla_{x_I} x_D = -\left[\nabla_{x_D} h(x_D(x_I,0),x_I,0)\right]^{-1} \nabla_{x_I} h(x_D(x_I,0),x_I,0).$$

Substituting this last expression in (3.4) we have:

$$\nabla_{x_I}\tilde{g}_i\,\tilde{y}_I = \nabla_x g_i \begin{bmatrix} -[\nabla_{x_D} h(x_D(x_I,0),x_I,0)]^{-1}\,\nabla_{x_I} h(x_D(x_I,0),x_I,0) \\ I \end{bmatrix} \tilde{y}_I,$$

and from (3.2) we see that at $(x_I^*,0)$

$$\nabla_{x_I}\tilde{g}_i\,\tilde{y}_I = \nabla_x g_i \begin{bmatrix} \tilde{y}_D \\ \tilde{y}_I \end{bmatrix} = \nabla_x g_i\,\tilde{y}_I. \qquad (3.5)$$

Thus, by (2.1) it follows that $\nabla_{x_I}\tilde{g}_i\,\tilde{y}_I > 0$.

In [13], Gauvin and Tolle established that the set of Kuhn-Tucker multipliers associated with a solution, x^*, of P(0) is non-empty, compact and convex if and only if MFCQ is satisfied at x^*. That result enables us to establish in Theorem 3.2, a necessary link between a directional derivative, with respect to the decision variable x_I, of the objective function at an optimal point and a directional derivative of the Lagrangian taken with respect to the parameter ϵ. It is this relationship which eventually leads to the upper and lower bounds on the directional derivative limit quotients which are derived in the next section.

We now obtain several perturbation results for problem $\tilde{P}(\epsilon)$. These *do not* depend on the variable-reduction derivation of $\tilde{P}(\epsilon)$ and are applicable to any inequality constrained problem having the indicated structure. Hence, unless otherwise stipulated, we assume in the following that a problem of form $\tilde{P}(\epsilon)$ is given, without reference to $P(\epsilon)$.

The next two theorems are crucial in obtaining the sharp bounds on the optimal value directional derivative limit quotient. They show that, at a local minimum where MFCQ holds, there exists a direction (in E^{n-p}) in which the directional derivative of the objective function yields that portion of the bound attributable to the constraint perturbation.

<u>**Theorem 3.2**</u>. If the conditions of MFCQ are satisfied for some $\bar{x}_I \varepsilon \tilde{S}(0)$, then, for any direction $z \varepsilon E^k$, there exists a vector $\bar{y} \varepsilon E^{n-p}$ satisfying:

i) $-\nabla\tilde{g}_i(\bar{x}_I,0)\ \bar{y} \leq \nabla_\varepsilon \tilde{g}_i(\bar{x}_I,0)z$ for $i\varepsilon\tilde{B}(0)$, and (3.6)

ii) $\nabla f(\bar{x}_I,0)\ \bar{y} = \max\limits_{\mu\varepsilon\tilde{K}(\bar{x}_I,0)} [-\mu'\nabla_\varepsilon \tilde{g}(\bar{x}_I,0)z]$. (3.7)

<u>Proof</u>. Given $z\varepsilon E^k$, consider the following linear program:

$$\max_\mu\ [-\mu'\nabla_\varepsilon \tilde{g}(\bar{x}_I,0)z]$$

$$\text{s.t. } \mu'\nabla\tilde{g}(\bar{x}_I,0) = \nabla f(\bar{x}_I,0)$$

$$\mu_i\tilde{g}_i(\bar{x}_I,0) = 0 \quad (i=1,\ldots,m)$$

$$\mu_i \geq 0 \quad (i=1,\ldots,m).$$

The dual of this program is given by:

$$\min_{y,v}\ \nabla\tilde{f}(\bar{x}_I,0)\ y$$

$$\text{s.t. } \nabla\tilde{g}_i(\bar{x}_I,0)\ y + \tilde{g}_i(\bar{x}_I,0)v_i \geq -\nabla_\varepsilon g_i(\bar{x}_I,0)z \quad (i=1,\ldots,m)$$

$$y\varepsilon E^{n-p},\ v_i \text{ unrestricted.}$$

Since MFCQ is assumed to hold at $(\bar{x}_I,0)$, from [13] we have that $\tilde{K}(\bar{x}_I,0)$ is nonempty, compact and convex. Thus, the primal problem is bounded and feasible. By the duality theorem of linear programming, the dual program has a solution, $(\bar{v},\bar{v})$, and hence there exists a vector $\bar{y}$ satisfying (3.6) and (3.7).

In the next two theorems we show first that, along any ray emanating from $\epsilon = 0$, $\tilde{P}(\epsilon)$ has points of feasibility near $\epsilon = 0$, and second, that the existence of feasible points is guaranteed not only along rays but in a full neighborhood of $\epsilon = 0$. In obtaining the following results associated with Problem $\tilde{P}(\epsilon)$, it is assumed that the analysis is confined to (x_I,ϵ) in N*.

Theorem 3.3. If MFCQ holds at $\bar{x}_I \epsilon \tilde{S}(0)$ then, for any unit vector $z \epsilon E^k$ and any $\delta > 0$, $\tilde{g}(\bar{x}_I + \beta(\bar{y} + \delta y_I),\beta z) > 0$ for β positive and sufficiently near zero, where $\bar{y}$ is any vector satisfying (3.6) and $\tilde{y}_I$ satisfies MFCQ.

Proof. Let z be any unit vector in E^k and consider first the case in which the constraint $g_i(x_I,\epsilon)$ 0 is binding at $(\bar{x}_I,0)$. Expanding $\tilde{g}_i(\bar{x}_I + \beta(\bar{y} + \delta y_I),\beta z)$ about the point $(\bar{x}_I,0)$ we obtain:

$$\begin{aligned}\tilde{g}_i(\bar{x}_I + \beta(\bar{y} + \delta y_I),\beta z) &= \beta\, \nabla_x \tilde{g}_i(\bar{x}_I + t\beta(\bar{y} + \delta y_I),\beta z)(\bar{y} + \delta y_I) \\ &\quad + \beta \quad \nabla_\epsilon \tilde{g}_i(\bar{x}_I,t'\beta z)\, z \\ &= \beta\ [\nabla_x \tilde{g}_i(\bar{x}_I + t\beta(\bar{y} + \delta y_I),\beta z)\bar{y} + \nabla_\epsilon \tilde{g}_i(\bar{x}_I,t'\beta z)\, z] \\ &\quad + \beta\delta \quad \nabla_x \tilde{g}_i(\bar{x}_I + t\beta(\bar{y} + \delta y_I),\beta z)\, y_I,\end{aligned}$$

where $t,t' \epsilon (0,1)$ and $t = t(\beta)$, $t' = t'(\beta)$.

Now, by (2.1), $\nabla_x g_i(\bar{x}_I,0)\tilde{y}_I = a_i > 0$. Thus, there exists $\beta' > 0$ such that for all $\beta \in [0,\beta']$,

$$\nabla_x \tilde{g}_i(\bar{x}_I + t\beta(\bar{y}+\delta\tilde{y}_I),\beta z)\tilde{y}_I \geq \frac{3a_i}{4} .$$

From (3.6) it follows that for β sufficiently small,

$$\nabla_x \tilde{g}_i(\bar{x}_I + t\beta(\bar{y}+\delta\tilde{y}_I),\beta z)\bar{y} + \nabla_\varepsilon \tilde{g}_i(\bar{x}_I, t'\beta z)z \geq -\frac{\delta a_i}{4} .$$

Thus, for β positive and near zero we have:

$$\tilde{g}_i(\bar{x}_I + \beta(\bar{y}+\delta\tilde{y}_I),\beta z) \geq \beta\left(-\frac{\delta a_i}{4}\right) + \beta\delta\left(\frac{3a_i}{4}\right) = \frac{\beta\delta a_i}{2} > 0 .$$

Finally, if $g_i(\bar{x}_I,0) > 0$, since each $\tilde{g}_i$ is jointly continuous in x and ε, it follows that, for any unit vector $z \in E^k$, and any $\delta > 0$, $g_i(\bar{x}_I + \beta(\bar{y}+\delta y_I),\beta z) > 0$ for β near zero.

Theorem 3.4. If MFCQ is satisfied at $\bar{x}_I \in \check{S}(0)$ and if $\varepsilon_k \to 0$, then, given any $\delta > 0$, there exists $\beta_{k_j} \to 0$ and a vector $\bar{y}$ such that $\tilde{g}(\bar{x}_I + \beta_{k_j}(\bar{y} + \delta y_I), \varepsilon_{k_j}) > 0$ for large j, where $\{\varepsilon_{k_j}\} \subseteq \{\varepsilon_k\}$, $\varepsilon_{k_j} \equiv ||\varepsilon_{k_j}||z_{k_j}$, $z_{k_j} \to \bar{z}$, $\bar{y}$ satisfies (3.6) for $z = \bar{z}$, and $\tilde{y}_I$ is given by MFCQ.

Proof. If $\{\varepsilon_{k_j}\} \subseteq \{\varepsilon_k\}$ and $\varepsilon_{k_j} = 0$ for every j, the conclusion follows for $\{\varepsilon_{k_j}\}$ and any $\bar{y}$ by taking $\beta_{k_j} = 0$ for every j. Suppose $\varepsilon_{k_j} \neq 0$ for every j. Define $\beta_{k_j} = ||\varepsilon_{k_j}||$ and $z_{k_j} = \varepsilon_{k_j} / ||\varepsilon_{k_j}||$. Then, relabeling z_{k_j} if necessary, we can assume there exists $\bar{z}$ such that $z_{k_j} \to \bar{z}$. Let $\bar{y}$ satisfy (3.6) for the vector $\bar{z}$. Then, from Theorem 3.3 and the continuity of g it follows that $\tilde{g}(x_I + \beta_{k_j}(\bar{y} + \delta\tilde{y}_I),\varepsilon_{k_j})$

$g(\bar{x}_I + \beta_{k_j}(\bar{y} + \delta\dot{y}_I), \beta_{k_j} z_{k_j}) > 0$ for large j.

By Theorem 3.4, the satisfaction of the Mangasarian-Fromovitz constraint qualification at a solution point $\bar{x}_I$ of $\tilde{P}(0)$ is enough to guarantee the existence of feasible points for $\tilde{P}(\varepsilon)$ near $\bar{x}_I$. One might suspect that there exist points feasible to $\tilde{P}(\varepsilon)$ which are also feasible to $\tilde{P}(0)$. This is indeed the case as the next theorem implies (see the statement immediately following the proof of Theorem 3.5). We shall need Theorem 3.5 in obtaining one of the key results in Theorem 4.3.

Theorem 3.5. Let $\beta_n \to 0^+$ in E^1, let z be any unit vector in E^k, and let $\delta > 0$. If $x_I^n \varepsilon \tilde{R}(\beta_n z)$, with $x_I^n \to \bar{x}_I \varepsilon \tilde{R}(0)$, and if the conditions of MFCQ are satisfied at $\bar{x}_I$, then $x_I^n + \beta_n(\bar{y} + \delta \tilde{y}_I) \varepsilon \tilde{R}(0)$ for n sufficiently large, where $\bar{y}$ satisfies (3.6) with z replaced by -z, and $\tilde{y}_I$ is given by the constraint qualification.

Proof. Consider first the case that $i \in \tilde{B}(0)$. Expanding $g_i(x_I^n + \beta_n(\bar{y} + \delta\tilde{y}_I),0)$ about the point $(x_I^n, \beta_n z)$, we obtain:

$$g_i(x_I^n + \beta_n(\bar{y} + \delta\tilde{y}_I),0) = g_i(x_I^n,\beta_n z) + \beta_n \nabla g_i(x_I^n + t\beta_n(\bar{y} + \delta\tilde{y}_I),0)(\bar{y} + \delta\tilde{y}_I)$$

$$- \beta_n \nabla_\epsilon \tilde{g}_i(x_I^n + \beta_n(\bar{y} + \delta\tilde{y}_I),t'\beta_n z)z,$$

where $t,t' \epsilon (0,1)$, $t = t(\beta_n)$, $t' = t'(\beta_n)$. If, for n large, $x_I^n + \beta_n(\bar{y} + \delta y_I) \notin \tilde{R}(0)$, since x_I^n is feasible for $\tilde{P}(\beta_n z)$, it must be that

$$\beta_n \nabla g_i(x_I^n + t\beta_n(\bar{y} + \delta y_I),0)(\bar{y} + \delta y_I) < \beta_n \nabla_\epsilon g_i(x_I^n + \beta_n(\bar{y} + \delta y_I),t'\beta_n z)z. \qquad (3.8)$$

Dividing by β_n in (3.10) and taking the limit as $n \to \infty$ we have

$$\nabla g_i(\bar{x}_I,0)(\bar{y} + \delta y_I) \leq \nabla_\epsilon g_i(\bar{x}_I,0)z.$$

But this contradicts (3.6) with z replaced by -z in (3.6), since $\delta > 0$ and by MFCQ $\nabla g_i(\bar{x}_I,0)y_I > 0$.

If, on the other hand, $i \notin \tilde{B}(0)$, $g_i(x_I^n + \beta_n(\bar{y} + \delta y_I),0) > 0$ for large n by the continuity of g_i and the fact that $x_I^n \to \bar{x}_I$ and $\beta_n \to 0$.

It may be interesting to note, that by taking $x_I^n = \bar{x}_I + \beta_n(\bar{y} + \delta\tilde{y}_I)$ for each n in the hypothesis of Theorem 3.5, then Theorems 3.4 and 3.5 together imply that $\tilde{R}(\epsilon)$ and $\tilde{R}(0)$ have points in common for ϵ near 0.

Let M be a closed subset of N* whose interior contains $(x_I^*,0)$ and denote by $\bar{P}(\epsilon)$ the Problem $\tilde{P}(\epsilon)$, where N* is replaced by M. The feasible region of $\bar{P}(\epsilon)$ will be denoted by $\bar{R}(\epsilon)$, the solution set by $\bar{S}(\epsilon)$, the optimal value by $\bar{f}^*(\epsilon)$, etc. Denote by M^0 the interior of M.

We now show that the optimal value functions $\bar{f}^*(\epsilon)$ and $f^*(\epsilon)$ of $\bar{P}(\epsilon)$ and $P(\epsilon)$, respectively, are continuous near $\epsilon = 0$ under the given assumptions. This is proved by the author under more general assumptions [8], the details being repeated here to make this paper complete. This result will be needed in the proof of Theorem 4.5. The continuity of $f^*(\epsilon)$ was also shown by Gauvin and Dubeau [12], under the same assumptions as those given here.

Lemma 3.6. If $\bar{R}(0)$ is nonempty and $\bar{R}(\epsilon)$ is uniformly compact for ϵ near zero, then $\bar{R}(\epsilon)$ is a closed mapping at $\epsilon = 0$.

Proof. Let $\epsilon_n \to 0$ in E^k, and (for n sufficiently large), let $x_I^n \epsilon \bar{R}(\epsilon_n)$. Thus, for n sufficiently large, $\tilde{g}(x_I^n,\epsilon_n) \geqq 0$ and $(x_I^n,\epsilon_n)\epsilon M$. By the uniform compactness of $\bar{R}(\epsilon)$, there exists a convergent subsequence $\{x_I^{n_j}\}$ of $\{x_I^n\}$ with $x_I^{n_j} \to \bar{x}_I$ for some $\bar{x}_I$ in the closure of $\cup\bar{R}(\epsilon)$ for ϵ near 0. But by the continuity of $\tilde{g}$, we must have $0 \leqq \lim_{j\to\infty} \tilde{g}(x_I^{n_j},\epsilon_{n_j}) = \tilde{g}(\bar{x}_I,0)$, and of course $(\bar{x}_I,0)\epsilon M$. Thus $\bar{x}_I \epsilon \bar{R}(0)$ and we have that $\bar{R}(\epsilon)$ is closed at $\epsilon = 0$.

Theorem 3.7. If $\bar{R}(0)$ is nonempty, $\bar{R}(\epsilon)$ is uniformly compact for ϵ near zero, and if there exists $\bar{x}_I \epsilon \bar{S}(0)$ such that $(\bar{x}_I,0) \epsilon M^0$ and the conditions of MFCQ hold at $\bar{x}_I$, then $\bar{f}^*(\epsilon)$ is continuous at $\epsilon = 0$.

Proof. Let $\epsilon_n \to 0$ in E^k be such that $\underline{\lim}_{\epsilon\to 0}\, \hat{f}^*(\epsilon) = \underline{\lim}_{n\to 0}\, \hat{f}^*(\epsilon_n)$. Clearly, since $\bar{R}(\epsilon_n) \neq \phi$ for n sufficiently large (Theorem 3.4), $\bar{S}(\epsilon_n) \neq \phi$ for large n. Hence, assuming n is large enough, there exists $x_I^n \epsilon \bar{S}(\epsilon_n)$. By the uniform compactness of $\bar{R}(\epsilon)$, the sequence $\{x_I^n\}$ admits a convergent subsequence $\{x_I^{n_j}\}$. Let $\bar{\bar{x}}_I$ denote the limit of that subsequence. From Lemma 3.6, $\bar{R}(\epsilon)$ is a closed mapping at $\epsilon = 0$, so $\bar{\bar{x}}_I \epsilon \bar{R}(0)$. Thus,

$$\underline{\lim}_{\varepsilon\to 0} \bar{f}^*(\varepsilon) = \lim_{j\to\infty} \bar{f}^*(\varepsilon_{n_j}) = \lim_{j\to\infty} \tilde{f}(x_I^{n_j},\varepsilon_{n_j}) = \tilde{f}(\bar{\bar{x}}_I,0) \geq \bar{f}^*(0),$$

and we see that $\bar{f}^*(\varepsilon)$ is lsc at $\varepsilon = 0$.

Now let $\delta > 0$, let $\tilde{y}_I$ be given by MFCQ for $\bar{x}_I$, and select $\varepsilon_n \to 0$ such that $\overline{\lim}_{\varepsilon\to 0} f^*(\varepsilon) = \overline{\lim}_{n\to\infty} \hat{f}^*(\varepsilon_n)$. From Theorem 3.4 we know $\beta_{n_j} \to 0$ and $\bar{y}$ such that $\bar{x}_I + \beta_{n_j}(\bar{y} + \delta\tilde{y}_I)\varepsilon\tilde{R}(\varepsilon_{n_j})$ for j large, where $\bar{y}$ satisfies (3.6) for some vector z. Hence,

$$\overline{\lim}_{\varepsilon\to 0} f^*(\varepsilon) = \overline{\lim}_{\varepsilon_{n_j}\to 0} \bar{f}^*(\varepsilon_{n_j}) \leq \overline{\lim}_{j\to\infty} \bar{f}(\bar{x}_I + \beta_{n_j}(\bar{y} + \delta\tilde{y}_I),\varepsilon_{n_j}) = \bar{f}(\bar{x},0) = \bar{f}^*(0).$$

Thus $\bar{f}^*(\varepsilon)$ is also usc at $\varepsilon = 0$ and we may conclude that $\bar{f}^*(\varepsilon)$ is continuous at $\varepsilon = 0$.

We should mention that the continuity of f* requires only the continuity of f in addition to the once (joint) continuous differentiability of the constraints.

The continuity of $\bar{f}^*(\varepsilon)$ at $\varepsilon = 0$ leads to a simple proof of the continuity of $f^*(\varepsilon)$, the optimal value of $P(\varepsilon)$. This is of intrinsic interest and will also be used in deriving the directional derivative limit quotient lower bound in the sequel. We first note the following result, the first part being an easy consequence of the continuity of the problem functions that is proved precisely analogously to Lemma 3.6. The proof that $f^*(\varepsilon)$ is lsc at $\varepsilon = 0$ precisely parallels the first part of the proof of Theorem 3.7 that shows that $\bar{f}^*(\varepsilon)$ is lsc.

<u>Lemma 3.8</u>. If R(0) is nonempty and $R(\varepsilon)$ is uniformly compact for ε near 0, then $R(\varepsilon)$ is a closed mapping and $f^*(\varepsilon)$ is lsc at $\varepsilon = 0$.

Theorem 3.9. If $R(0)$ is nonempty, $R(\varepsilon)$ is uniformly compact for ε near 0, and if MFCQ holds at some $x^*\varepsilon S(0)$, then $f^*(\varepsilon)$ is continuous at $\varepsilon = 0$.

Proof. We eliminate the equalities of $P(\varepsilon)$ at x^* for (x_I,ε) in a neighborhood N^* of $(x_I^*,0)$, where $x^* = (x_D^*,x_I^*)$, using the previously defined variable reduction transformation, constructing problems of the form $\tilde{P}(\varepsilon)$ and $\bar{P}(\varepsilon)$. We know that $x_I^*\varepsilon\tilde{S}(0)$ and that MFCQ holds at x_I^* (Lemma 3.1). Also, the uniform compactness of $R(\varepsilon)$ near $\varepsilon = 0$ implies the uniform compactness of $\bar{R}(\varepsilon)$ near $\varepsilon = 0$.

Clearly, $f^*(\varepsilon) \leq \tilde{f}^*(\varepsilon) \leq \bar{f}^*(\varepsilon)$ and since $f^*(0) = f(x^*,0) = \tilde{f}(x_I^*,0)$, we conclude that $\bar{f}^*(0) = \tilde{f}^*(0) = f^*(0)$, which also implies that $x_I^*\varepsilon\bar{S}(0)$. The assumptions of Theorem 3.7 are satisfied, hence $\bar{f}^*(\varepsilon)$ is continuous at 0. These relationships imply that $\overline{\lim}_{\varepsilon\to 0} f^*(\varepsilon) \leq \overline{\lim}_{\varepsilon\to 0} \bar{f}^*(\varepsilon) = \bar{f}^*(0) = f^*(0)$; i.e., $f^*(\varepsilon)$ is usc at 0. Since $f^*(\varepsilon)$ is also lsc at $\varepsilon = 0$ (Lemma 3.8), the conclusion follows.

4. Bounds on the Parametric Variation of the Optimal Value Function

In this section we are concerned with the directional derivative of the optimal value function for $P(\varepsilon)$. We first derive upper and lower bounds on the directional derivative limit quotient of $\tilde{f}^*(\varepsilon)$ for $\tilde{P}(\varepsilon)$ and then obtain the corresponding bounds for $P(\varepsilon)$. These results extend the work of Gauvin and Tolle [13], who obtained the analogous results for the case in which the perturbation is restricted to the right-hand side of the constraints.

As above, we will, without loss of generality, focus attention on the parameter value $\varepsilon = 0$. For $z\varepsilon E^k$, the directional derivative of $\tilde{f}^*(\varepsilon)$ at $\varepsilon = 0$ in the direction z is defined to be:

$$D_z\tilde{f}^*(0) = \lim_{\beta\to 0^+} \frac{\tilde{f}^*(\beta z) - \tilde{f}^*(0)}{\beta}, \tag{4.1}$$

providing that the limit exists.

<u>Theorem 4.1</u>. If, for $\tilde{P}(\varepsilon)$, MFCQ holds for some $\bar{x}_I \varepsilon \tilde{S}(0)$, then, for any direction $z\varepsilon E^k$,

$$\limsup_{\beta\to 0^+} \frac{\tilde{f}^*(\beta z) - \tilde{f}^*(0)}{\beta} \leq \max_{\mu\varepsilon\tilde{K}(\bar{x}_I,0)} \nabla_\varepsilon \tilde{L}(\bar{x}_I,\mu,0)z. \tag{4.2}$$

<u>Proof</u>. Let β satisfy the conditions of Theorem 3.3, let $\delta > 0$ and $\tilde{y}_I$ be the vector given by the constraint qualification, and let $\bar{y}$ satisfy eqs. (3.6) and (3.7). Then, for any $z\varepsilon E^k$, $\bar{x}_I + \beta(\bar{y} + \delta\tilde{y}_I)\varepsilon\tilde{R}(\beta z)$ for β near 0, so that

$$\lim_{\beta\to 0^+} \sup \frac{\tilde{f}^*(\beta z)-\tilde{f}^*(0)}{\beta} \leq \lim_{\beta\to 0^+} \sup \frac{\tilde{f}(\bar{x}_I+\beta(\bar{y}+\delta\tilde{y}_I),\beta z) - \tilde{f}(\bar{x}_I,0)}{\beta} = \frac{d\tilde{f}}{d\beta}(\bar{x}_I,0)$$

$$= \nabla\tilde{f}(\bar{x}_I,0)(\bar{y} + \delta\tilde{y}_I) + \nabla_\varepsilon\tilde{f}(\bar{x}_I,0)z.$$

Since this inequality is satisfied for arbitrary $\delta > 0$ we can take the limit as $\delta \to 0$ and obtain:

$$\lim_{\beta\to 0^+} \sup \frac{\tilde{f}^*(\beta z) - \tilde{f}^*(0)}{\beta} \leq \nabla\tilde{f}(\bar{x}_I,0)\bar{y} + \nabla_\varepsilon\tilde{f}(\bar{x}_I,0)z.$$

The conclusion now follows by applying (3.7):

$$\lim_{\beta\to 0^+} \sup \frac{\tilde{f}^*(\beta z) - \tilde{f}^*(0)}{\beta} < \max_{\mu\varepsilon\tilde{K}(x_I^*,0)} [-\mu'\nabla_\varepsilon\tilde{g}(\bar{x}_I,0) + \nabla_\varepsilon\tilde{f}(\bar{x}_I,0)]z$$

$$= \max_{\mu\varepsilon\tilde{K}(x_I^*,0)} \nabla_\varepsilon\tilde{L}(\bar{x}_I,\mu,0)z.$$

Corollary 4.2. Under the hypotheses of the previous theorem, if MFCQ holds at each point $x_I\varepsilon\tilde{S}(0)$, then

$$\lim_{\beta\to 0^+} \sup \frac{\tilde{f}^*(\beta z) - \tilde{f}^*(0)}{\beta} \leq \inf_{x_I\varepsilon\tilde{S}(0)} \max_{\mu\varepsilon\tilde{K}(x_I,0)} \nabla_\varepsilon\tilde{L}(x_I,\mu,0)z. \quad (4.3)$$

Proof. The result follows directly by applying the previous theorem at each point of $\tilde{S}(0)$.

To obtain a lower bound on the directional derivative limit quotient, we use MFCQ and the following result which is well known and follows easily from the results obtained in the last section.

Lemma 4.3. If $\bar{R}(0) \neq \phi$, $\bar{R}(\varepsilon)$ is uniformly compact near $\varepsilon = 0$, and $\bar{f}^*(\varepsilon)$ is continuous at $\varepsilon = 0$, then $\bar{S}(\varepsilon)$ is closed at $\varepsilon = 0$.

Proof. Suppose $\varepsilon_n \to 0$ as $n \to \infty$ and suppose $x_I^n\varepsilon\bar{S}(\varepsilon_n)$ is such that $x_I^n \to x_I^*$. By Lemma 3.6, $\bar{R}(\varepsilon)$ is closed at $\varepsilon = 0$, so $x_I^*\varepsilon\bar{R}(0)$. Since $\bar{f}^*(\varepsilon)$ is continuous at $\varepsilon = 0$, it follows that $\lim_{n\to\infty} \bar{f}^*(\varepsilon_n) = \lim_{n\to\infty} \tilde{f}(x_I^n,\varepsilon_n) = \tilde{f}(x_I^*,0) = \bar{f}^*(0)$, hence $x_I^*\varepsilon\bar{S}(0)$.

The next lemma is an immediate consequence of this result and Theorem 3.7.

Lemma 4.4. Suppose $\epsilon_n \to 0$ and the assumptions of Theorem 3.7 hold. Then, for n large, there exists $x_I^n \epsilon \bar{S}(\epsilon_n)$ and all limit points of $\{x_I^n\}$ are in $\bar{S}(0)$.

Proof. The fact that $\bar{S}(\epsilon) \neq 0$ for near 0 follows from the fact that $\bar{R}(\epsilon_n) \neq \phi$ (Theorem 3.4) and compact for n large (since $\bar{R}(\epsilon)$ is uniformly compact for ϵ near 0) and since $\tilde{f}(x_I, \epsilon_n)$ is continuous in N*. From Theorem 3.7, we know that $\bar{f}^*(\epsilon)$ is continuous at $\epsilon = 0$. The conclusion then follows from the previous lemma.

Definition 4.5. For any given vector $z \epsilon E^k$, an infimal sequence x_I^n of the directional derivative limit quotient of $\bar{f}^*(\epsilon)$ is defined as $\{x_I^n\}$ such that $x_I^n \epsilon \bar{S}(\beta_n z)$ and

$$\liminf_{\beta \to 0^+} \frac{\bar{f}^*(\beta z) - \bar{f}^*(0)}{\beta} = \lim_{n \to \infty} \frac{\tilde{f}(x_I^n, \beta_n z) - \tilde{f}(x_I^*, 0)}{\beta_n}.$$

Theorem 4.6. Suppose z is any given vector in E^k. Suppose the assumptions of Theorem 3.7 are satisfied and suppose that $\bar{x}_I$ is a limit point of an infimal sequence $\{x_I^n\}$ as defined in Definition 4.5, relative to the given vector z. Then,

$$\liminf_{\beta \to 0^+} \frac{\bar{f}^*(\beta z) - \bar{f}^*(0)}{\beta} \gtrapprox \min_{\mu \epsilon \bar{K}(\bar{x}_I, C)} \nabla_\epsilon \tilde{L}(\bar{x}_I, \mu, 0) z. \tag{4.4}$$

Proof. Let $\beta_n \to 0^+$. We already know from Lemma 4.3 that there exists $x_I^n \epsilon \bar{S}(\beta_n z)$ for n large, and all limit points of $\{x_I^n\}$ are in $\bar{S}(0)$. Also, by definition, an infimal sequence as defined above always exists and there must exist at least one limit point in $\bar{S}(0)$ of this sequence.

Relabeling if necessary, our assumptions allow us to conclude a bit more, i.e., that $x_I^n \to \bar{x}_I \varepsilon \bar{S}(0) \cap N^*$, where $\{x_I^n\}$ is an infimal sequence relative to the given vector z.

Since MFCQ holds at $\bar{x}_I$, Theorem 3.5 assures that $x_I^n + \beta_n(\bar{y}+\delta\tilde{y}_I)\varepsilon\tilde{R}(0)$ for n sufficiently large. It follows that

$$\liminf_{\beta\to 0^+} \frac{\bar{f}^*(\beta z) - \bar{f}^*(0)}{\beta} = \lim_{n\to\infty} \frac{\tilde{f}(x_I^n,\beta_n z) - \tilde{f}(\bar{x}_I,0)}{n}$$

$$\geqq \lim_{n\to\infty} \frac{\tilde{f}(x_I^n,\beta_n z) - \tilde{f}(x_I^n+\beta_n(\bar{y}+\delta\tilde{y}_I),0)}{\beta_n}$$

$$= \lim_{n\to\infty} [-\nabla\tilde{f}(\alpha_n)(\bar{y}+\delta\tilde{y}_I) + \nabla_\varepsilon\tilde{f}(\alpha_n)z]$$

by the mean value theorem, where α_n is the usual convex combination (in $E^{n-p}\times E^k$) of the two arguments in the preceding quotient. Thus,

$$\liminf_{\beta\to 0^+} \frac{\bar{f}^*(\beta z) - \bar{f}^*(0)}{\beta} \geqq \lim_{n\to\infty} \nabla_\varepsilon\tilde{f}(\alpha_n)z - \nabla\tilde{f}(\alpha_n)(\bar{y}+\delta\tilde{y}_I)$$

$$= \nabla_\varepsilon\tilde{f}(\bar{x}_I,0)z - \nabla\tilde{f}(\bar{x}_I,0)(\bar{y}+\delta\tilde{y}_I).$$

Using Theorem 3.2 and noting that δ was chosen as any positive value, we conclude that

$$\liminf_{\beta\to 0^+} \frac{\bar{f}^*(\beta z) - \bar{f}^*(0)}{\beta} \geqq \nabla_\varepsilon\tilde{f}(\bar{x}_I,0)z - \max_{\mu\varepsilon\bar{K}(\bar{x}_I,0)} [\mu'\nabla_\varepsilon\tilde{g}(\bar{x}_I,0)z]$$

$$= \min_{\mu\varepsilon\bar{K}(\bar{x}_I,0)} \nabla_\varepsilon\tilde{L}(\bar{x}_I,\mu,0)z.$$

<u>Corollary 4.7</u>. Under the hypotheses of the previous theorem

$$\liminf_{\beta\to 0^+} \frac{\bar{f}^*(\beta z) - \bar{f}^*(0)}{\beta} \geqq \inf_{x_I\varepsilon\bar{S}(0)} \min_{\mu\varepsilon\bar{K}(x_I,0)} \nabla_\varepsilon\tilde{L}(x_I,\mu,0)z. \quad (4.5)$$

By the reduction of variables that was applied earlier, in a neighborhood of $(x_I^*,0)$, with $x = (x_D(x_I,\varepsilon),x_I)$,

$$\begin{aligned} L(x,\mu,\omega,\varepsilon) &= f(x,\varepsilon) - \mu' g(x,\varepsilon) + \omega' h(x,\varepsilon) \\ &= f(x_D(x_I,\varepsilon),x_I,\varepsilon) - \mu' g(x_D(x_I,\varepsilon),x_I,\varepsilon) + \omega' h(x_D(x_I,\varepsilon),x_I,\varepsilon) \\ &= \tilde{f}(x_I,\varepsilon) - \mu' \tilde{g}(x_I,\varepsilon) = \tilde{L}(x_I,\mu,\varepsilon), \end{aligned}$$

with $f(x,\varepsilon) \equiv \tilde{f}(x_I,\varepsilon)$, $g(x,\varepsilon) \equiv \tilde{g}(x_I,\varepsilon)$, and $h(x,\varepsilon) \equiv \tilde{h}(x_I,\varepsilon) \equiv 0$. Thus $L(x,\mu,\omega,\varepsilon) \equiv \tilde{L}(x_I,\mu,\varepsilon)$ in a neighborhood of $(x_D(x_I^*,0),x_I^*,0) = (x^*,0)$ and, with ω determined by $\omega' = -(\nabla_{x_D} f - \mu' \nabla_{x_D} g)[\nabla_{x_D} h]^{-1}$, it follows easily that $\nabla_\varepsilon \tilde{L} = \nabla_\varepsilon L$ and the linear program appearing in the proof of Theorem 3.2 and involved in the preceding bounds can readily be formulated analogously as a locally equivalent problem in terms of $L(x,\mu,\omega,\varepsilon)$.

We now utilize the above results obtained for $\tilde{P}(\varepsilon)$ and $\bar{P}(\varepsilon)$ to obtain bounds for the optimal value directional derivative quotient of $P(\varepsilon)$.

Theorem 4.8. If MFCQ holds at some $x^* \varepsilon S(0)$, then for any direction $z \varepsilon E^k$,

$$\limsup_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \leqq \max_{(\mu,\omega)\varepsilon K(x^*,0)} \nabla_\varepsilon L(x^*,\mu,\omega,0) z$$

Proof. Apply the variable reduction transformation at $x^* = (x_D^*,x_I^*)$ as in the previous construction to obtain a problem of the form $\tilde{P}(\varepsilon)$, defined for (x_I,ε) in a neighborhood N^* of $(x_I^*,0)$. Since $f^*(\varepsilon) \leqq \tilde{f}^*(\varepsilon)$ and $f^*(0) = \tilde{f}^*(0)$, we have that

$$\limsup_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \leqq \limsup_{\beta \to 0^+} \frac{\tilde{f}^*(\beta z) - \tilde{f}^*(0)}{\beta}$$

and the conclusion is an immediate consequence of Theorem 4.1, having expressed the bound in (4.2) in terms of the original variables by way of the variable reduction transformation.

<u>Theorem 4.9</u>. If $R(0) \neq \phi$, $R(\varepsilon)$ is uniformly compact near $\varepsilon = 0$, and MFCQ holds for each $x \in S(0)$, then for any direction $z \in E^k$,

$$\liminf_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \geq \min_{(\mu,\omega) \in K(x^*,0)} \nabla_\varepsilon L(x^*,\mu,\omega,0)z$$

holds for some $x^* \in S(0)$.

<u>Proof</u>. Given any $z \in E^n$, consider $x^n \in S(\beta_n z)$ such that

$$\liminf_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} = \lim_{n \to \infty} \frac{f(x^n,\beta_n z) - f(x^*,0)}{\beta_n}.$$

Since $R(\varepsilon)$ is uniformly compact, there exists a subsequence, which we again denote by $\{x^n\}$, and a vector x^* such that $x_n \to x^*$. By Lemma 3.8, $R(\varepsilon)$ is closed and, by Theorem 3.9, $f^*(\varepsilon)$ is continuous at $\varepsilon = 0$. It follows (as in the proof of Lemma 4.3) that $S(\varepsilon)$ is closed at 0, so $x^* \in S(0)$.

We now apply the variable reduction transformation at $x^* = (x_D^*, x_I^*)$, following the usual construction, and obtain a problem of the form $\tilde{P}(\varepsilon)$, defined for (x_I,ε) in a neighborhood N^* of $(x_I^*,0)$. We also define the reduced problem $\bar{P}(\varepsilon)$, i.e., problem $\tilde{P}(\varepsilon)$ with M, a closed subset of N^* whose interior contains $(x_I^*,0)$, replacing N^*.

Noting that in N^*, $f^*(\beta_n z) = \bar{f}^*(\beta_n z)$ and $f^*(0) = \bar{f}^*(0)$, it is easily verified that the assumptions of Theorem 4.6 are satisfied and we also have

$$\liminf_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} = \liminf_{\beta \to 0^+} \frac{\bar{f}^*(\beta z) - \bar{f}^*(0)}{\beta},$$

from which the conclusion follows, expressing the right hand side of (4.4) in terms of the original variables via the variable reduction transformation.

Clearly Corollaries 4.2 and 4.7 immediately extend to $f^*(\epsilon)$ as well, using these results. Thus, all of the results obtained above for $\tilde{P}(\epsilon)$ and $\bar{P}(\epsilon)$ can be immediately generalized to $P(\epsilon)$. For completeness, we state these results as the next theorem.

Theorem 4.10. If, for $P(\epsilon)$, $R(0)$ is nonempty and MFCQ holds at each $x \epsilon S(0)$, then for any unit vector $z \epsilon E^k$,

$$\limsup_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \leq \inf_{x \epsilon S(0)} \max_{(\mu,\omega) \epsilon K(x,0)} \nabla_\epsilon L(x,\mu,\omega,0) z, \quad (4.6)$$

and if $R(\epsilon)$ is uniformly compact for ϵ near $\epsilon = 0$, then

$$\liminf_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \geq \inf_{x \epsilon S(0)} \min_{(\mu,\omega) \epsilon K(x,0)} \nabla_\epsilon L(x,\mu,\omega,0) z. \quad (4.7)$$

Moreover, we are able to obtain the existence of the directional derivative of f^* at $\epsilon = 0$ by assuming, as Gauvin and Tolle [13] did for right-hand side programs, the linear independence of the binding constraint gradients at each point $x^* \epsilon S(0)$.

Corollary 4.11. Assume $R(0)$ is nonempty and $R(\epsilon)$ is uniformly compact near $\epsilon = 0$ If the gradients, taken with respect to x, of the constraints binding at x^* are linearly independent for each $x^* \epsilon S(0)$, then for any unit vector $z \epsilon E^k$, $D_z f^*(0)$ exists and is given by

$$D_z f^*(0) = \inf_{x \epsilon S(0)} \nabla_\epsilon L(x,\mu(x),\omega(x),0) z,$$

where $(\mu(x),\omega(x))$ is the unique multiplier vector associated with x.

Proof. At any point $x^* \epsilon S(0)$, the linear independence of the binding constraint gradients implies the uniqueness of the Kuhn-Tucker multipliers corresponding to x^*. Inequalities (4.6) and (4.7) now combine to yield the desired result.

Note that in Corollary 4.11 if $P(\epsilon)$ contains no inequality constraints, we could replace inf by min since μ would not appear and $\omega' = -\nabla_{x_D} f \, \nabla_{x_D} h^{-1}$ which is continuous in x, making $\nabla_\epsilon Lz$ a continuous function of x minimized over S(0), a compact set.

We may also show that two of the observations made by Gauvin and Tolle [13] about $D_z f^*(0)$ for right-hand side programs apply to $P(\epsilon)$ as well. First if $D_z f^*(0) = - D_{-z} f^*(0)$, then

$$\inf_{x \epsilon S(0)} \max_{(\mu,\omega)\epsilon K(x,0)} \nabla_\epsilon L(x,\mu,\omega,0)z = \sup_{x\epsilon S(0)} \min_{(\mu,\omega)\epsilon K(x,0)} \nabla_\epsilon L(x,\mu,\omega,0)z. \quad (4.8)$$

Thus, if, for all unit vectors $z \epsilon E^k$, $D_z f^*(0) = - D_{-z} f^*(0)$ and

$$D_z f^*(0) = \inf_{x\epsilon S(0)} \max_{(\mu,\omega)\epsilon K(x,0)} \nabla_\epsilon L(x,\mu,\omega,0)z, \quad (4.9)$$

then (4.8) provides a necessary condition for the existence of $\nabla_\epsilon f^*(0)$. In addition, if (4.8) holds for every unit vector $z \epsilon E^k$ and if $x^* \epsilon S(0)$ is the unique solution of P(0), then its associated Kuhn-Tucker multiplier vector is unique.

We next apply the results derived above to a particular class of programs. We show in the next theorem that if $P(\epsilon)$ is a convex program in x for ϵ near $\epsilon = 0$, i.e., if $f(x,\epsilon)$ and $-g_i(x,\epsilon)$,

$i=1,\ldots,m$, are convex and if $h_j(x,\varepsilon)$, $j=1,\ldots,p$, are affine in x, then $D_z f^*(0)$ exists and is given by (4.9). To prove this result, we will restrict our attention to convex programs of the form $\tilde{P}(\varepsilon)$, with (x_I,ε) no longer constrained to be in a specified neighborhood N*. We are able to do this since the functions $h_j(x,\varepsilon)$ are assumed to be affine in x and f(x) and the $-g_i(x,\varepsilon)$ are taken to be convex in x, from which it easily follows that the variable reduction transformation applies globally, and further, $\tilde{f}(x_I,\varepsilon)$ and the $-\tilde{g}_i(x_I,\varepsilon)$ are convex in x_I for $i=1,\ldots,m$.

Theorem 4.12. In $P(\varepsilon)$, let $f(x,\varepsilon)$ and $-g_i(x,\varepsilon)$, $i = 1,\ldots,m$ be convex and let $h_j(x,\varepsilon)$, $j = 1,\ldots,p$ be affine in x. If R(0) is nonempty, $R(\varepsilon)$ is uniformly compact near $\varepsilon = 0$, and MFCQ holds for each $x^*\varepsilon S(0)$, then, for any unit vector $z\varepsilon E^k$,

$$D_z f^*(0) = \inf_{x\varepsilon S(0)} \max_{(\mu,\omega)\varepsilon K(x,0)} \nabla_\varepsilon L(x,\mu,\omega,0)z. \qquad (4.10)$$

Proof. Without loss of generality, as indicated, we will prove this result for $\tilde{P}(\varepsilon)$, with the set N* not present. For convenience, the notation is somewhat simplified by dropping the subscript I in the argument. Note that the assumptions imply that $\tilde{R}(\varepsilon) \equiv \{x|\tilde{g}_i(x,\varepsilon) \geqq 0,\ i=1,\ldots,m\}$ is a convex uniformly compact set near $\varepsilon = 0$.

Let $x^*\varepsilon\tilde{S}(0)$ and $x_n\varepsilon\tilde{S}(\beta_n z)$ with $\beta_n \to 0^+$ in such a way that

$$\liminf_{\beta\to 0^+} \frac{\tilde{f}^*(\beta z) - \tilde{f}^*(0)}{\beta} = \lim_{n\to\infty} \frac{\tilde{f}(x_n,\beta_n z) - \tilde{f}(x^*,0)}{\beta_n},$$

and $x_n \to x^*$ as in the proof of Theorem 4.9. For all $\mu^*\varepsilon\tilde{K}(x^*,0)$,

$$\tilde{L}(x_n,\mu^*,\beta_n z) = \tilde{f}(x_n,\beta_n z) - \mu^{*\prime}\tilde{g}(x_n,\beta_n z) \leq \tilde{f}(x_n,\beta_n z),$$

where the inequality follows from the non-negativity of both μ^* and $\tilde{g}(x_n,\beta_n z)$. Thus, since $\tilde{L}(x^*,\mu^*,0) = f(x^*,0)$,

$$\lim_{n\to\infty} \frac{\tilde{f}(x_n,\beta_n z) - \tilde{f}(x^*,0)}{\beta_n} \geq \lim_{n\to\infty} \frac{\tilde{L}(x_n,\mu^*,\beta_n z) - \tilde{L}(x^*,\mu^*,0)}{\beta_n}.$$

Now, as a result of the Kuhn-Tucker conditions and the convexity assumptions, x^* is a global minimizer of $\tilde{L}(x,\mu^*,0)$, so

$$\lim_{n\to\infty} \frac{\tilde{L}(x_n,\mu^*,\beta_n z) - \tilde{L}(x^*,\mu^*,0)}{\beta_n} \geq \lim_{n\to\infty} \frac{\tilde{L}(x_n,\mu^*,\beta_n z) - \tilde{L}(x_n,\mu^*,0)}{\beta_n}$$

$$= \lim_{n\to\infty} \frac{\tilde{L}(x_n,\mu^*,0) + \beta_n \nabla_\varepsilon \tilde{L}(x_n,\mu^*,t\beta_n z)z - \tilde{L}(x_n,\mu^*,0)}{\beta_n}$$

by the mean value theorem, where $t\varepsilon(0,1)$. Thus

$$\lim_{n\to\infty} \frac{\tilde{f}(x_n,\beta_n z) - \tilde{f}(x^*,0)}{\beta_n} \geq \lim_{n\to\infty} \nabla_\varepsilon \tilde{L}(x_n,\mu^*,t\beta_n z)z,$$

and, passing to the limit on the right, we are able to conclude that

$$\lim_{n\to\infty} \frac{\tilde{f}(x_n,\beta_n z) - \tilde{f}(x^*,0)}{\beta_n} \geq \nabla_\varepsilon \tilde{L}(x^*,\mu^*,0)z. \tag{4.11}$$

Thus, for some $x^*\varepsilon\tilde{S}(0)$, since (4.11) holds for each $\mu^*\varepsilon\tilde{K}(x^*,0)$, and recalling from [13] that $\tilde{K}(x^*,0)$ is compact,

$$\liminf_{\beta\to 0^+} \frac{\tilde{f}^*(\beta z) - \tilde{f}^*(0)}{\beta} \geq \max_{\mu\varepsilon\tilde{K}(x^*,0)} \nabla_\varepsilon \tilde{L}(x^*,\mu,0)z,$$

from which we see that

$$\liminf_{\beta\to 0^+} \frac{\tilde{f}^*(\beta z) - \tilde{f}^*(0)}{\beta} \geq \inf_{x\varepsilon\tilde{S}(0)} \max_{\mu\varepsilon\tilde{K}(x^*,0)} \nabla_\varepsilon \tilde{L}(x,\mu,0)z.$$

Combining this result with that obtained in Corollary 4.2 we conclude that

$$D_z f^*(0) = \inf_{x\varepsilon\tilde{S}(0)} \max_{\mu\varepsilon\tilde{K}(x,0)} \nabla_\varepsilon \tilde{L}(x,\mu,0)z.$$

For convex $P(\varepsilon)$, (4.10) now follows by an inversion of the reduction of variables process applied to yield $\tilde{P}(\varepsilon)$.

5. Example

We use the example stated below to demonstrate some of the theoretical results obtained in the previous sections. For the given problem we show that the conditions of MFCQ hold at every point in $S(\varepsilon)$ and we give the form of the vector satisfying the constraint qualification. We obtain the form of the vector satisfying (2.1) and (2.2) for $\tilde{P}(\varepsilon)$, and show that the bounds stated in (4.6) and (4.7) are attained.

Consider the program

$$\begin{aligned} &\min\ \varepsilon x_1 && P(\varepsilon) \\ &\text{s.t. } g(x,\varepsilon) = -(x_1-\varepsilon)^2 - (x_2-2)^2 + 4 \geq 0 \\ &\qquad h(x,\varepsilon) = -x_1 + x_2 + \varepsilon = 0. \end{aligned}$$

The solution of this program is easily determined to be $x_1^* = x_2^* + \varepsilon$ with

$$x_2^* = \begin{cases} 0 & \varepsilon>0 \\ 2 & \varepsilon<0 \end{cases}$$, and if $\varepsilon = 0$, x_2^* can be any value in the interval [0,2]. (5.1)

Applying the reduction of variables technique outlined earlier, with $x_D = x_1$ and $x_I = x_2$, $P(\varepsilon)$ is transformed into the equivalent program

$$\begin{aligned} &\min \quad \varepsilon(x_2+\varepsilon) && \tilde{P}(\varepsilon) \\ &\text{s.t. } \tilde{g}(x_2,\varepsilon) = -x_2^2 - (x_2-2)^2 + 4 \geq 0 \end{aligned}$$

whose solution is given by (5.1).

For both $P(\varepsilon)$ and $\tilde{P}(\varepsilon)$, the optimal value function can be written as

$$f^*(\varepsilon) = \begin{cases} \varepsilon^2 & \varepsilon \geq 0 \\ \varepsilon^2+2\varepsilon & \varepsilon < 0 \end{cases} . \qquad (5.2)$$

We see that f^* is continuous for all values of ε, but it is not differentiable at $\varepsilon = 0$. It does, however, have directional derivatives at $\varepsilon = 0$ which are given by

$$D_z f^*(0) = \begin{cases} 0 & z = 1 \\ -2 & z = -1 \end{cases} . \qquad (5.3)$$

To illustrate Lemma 3.1, we first determine the general form of the vector, $\tilde{y}$, associated with points $x \in S(\varepsilon)$ at which MFCQ is satisfied. The constraint gradients of $P(\varepsilon)$ are

$$\nabla g(x,\varepsilon) = [-2(x_1-\varepsilon), -2(x_2-2)], \text{ and}$$

$$\nabla h(x,\varepsilon) = [-1,1].$$

Applying (2.1) and (2.2) at a point $x^* = (x_1^*, x_2^*) \varepsilon S(\varepsilon)$, with $\tilde{y} = (y_1, y_2)$, we require that $\nabla g(x^*,\varepsilon)\tilde{y} = -2(x_1^*-\varepsilon)y_1 - 2(x_2^*-2)y_2 > 0$ if $g(x^*,0) = 0$, and

$$\nabla h(x^*,\varepsilon)\ \tilde{y} = -y_1 + y_2 = 0.$$

Thus, for any value of ε, since $g(x,\varepsilon)$ is binding only if $x_2^* = 0,2$, $\tilde{y}$ can have the form

$$\tilde{y} = \begin{cases} (a,a) & x_2^* = 0 \\ (b,b) & 0 < x_2^* < 2\ , \\ (c,c) & x_2^* = 2 \end{cases} \tag{5.4}$$

for any real numbers a,b,c with $a > 0$, $b \neq 0$, and $c < 0$. We can also conclude that *MFCQ holds at* every solution of $P(\varepsilon)$.

In a similar fashion, we see that, for $\tilde{P}(\varepsilon)$,

$$\nabla\tilde{g}(x_2,\varepsilon) = -2x_2 - 2(x_2-2),$$

so applying (2.1) we find that the vector $\tilde{y}_I$ in the reduced program takes the same form as the second component of $\tilde{y}$ in (5.4).

Now

$$\nabla L(x,\mu,\omega,\varepsilon) = [\varepsilon + 2\mu(x_1-\varepsilon) - \omega,\ 2\mu(x_2-2) + \omega],$$

so that at a solution $x^* \varepsilon S(0)$ we must have $2\mu x_1^* - \omega = 0$ for $(\mu,\omega)\varepsilon K(x^*,0)$. Then

$$\nabla_\varepsilon L(x^*,\mu,\omega,0) = x_1^* ,$$

and, with $S(0) = \{x\varepsilon E^2\colon x_1=x_2,\ x_2\varepsilon[0,2]\}$,

$$\min_{x\varepsilon S(0)} \max_{(\mu,\omega)\varepsilon K(x,0)} \nabla_\varepsilon L(x,\mu,\omega,0)z = \begin{cases} 0 & z = 1 \\ -2 & z = -1 \end{cases} . \tag{5.5}$$

Comparing (5.3) with (5.5) we see that (4.6) holds with equality.

Now, considering inequality (4.7), we first note that for any neighborhood $N(0)$ of $0\varepsilon E^1$, the closure of the set $\{x\varepsilon E^2\colon x = (x_2+\varepsilon,\ x_2),\ x_2\varepsilon[0,2],\ \varepsilon \text{ in } N(0)\}$ is compact so $R(\varepsilon)$ is uniformly compact for ε near $\varepsilon = 0$. We calculate

$$\min_{x\varepsilon S(0)} \min_{(\mu,\omega)\varepsilon K(x,0)} \nabla_\varepsilon L(x,\mu,\omega,0)z = \begin{cases} 0 & z = 1 \\ -2 & z = -1 \end{cases},$$

and find that (4.7) also holds with equality.

The above results could have been anticipated from (4.10), since the conditions of Theorem 4.12 hold for this example.

An example is given in [13] which illustrates that (4.6) and (4.7) need not hold with equality.

6. Related Results

Inspection of the derivation of (4.6) and (4.7) reveals that the bounding term in these expressions, namely $\nabla_\varepsilon L(x,\mu,\omega,0)z$, can be viewed as the sum of two distinct expressions, one resulting from the variation of the objective function of $P(\varepsilon)$ with respect to the parameter, the other deriving from the dependence of the region of feasibility on the parameter. The first of these terms is $\nabla_\varepsilon f(x,0)z$ and is easily seen to result directly from the manipulation of the limit quotients in the proofs of Theorems 4.1 and 4.6. The second component, $[-\mu'\nabla_\varepsilon' g(x,0) + \omega'\nabla_\varepsilon' h(x,\varepsilon)]z$, results from the assumption that MFCQ holds at points of S(0). The conditions of MFCQ are invoked to enable us to conclude (3.7), as well as the existence of points feasible to $P(\varepsilon)$ in a neighborhood of $\varepsilon = 0$. Having made these observations, we are now able to discuss the relationships between the bounds provided here and results previously obtained by others. As we shall see, in particular instances in which the directional derivative of f^* is shown to exist, it is expressed as either a function of $\nabla_\varepsilon f$ or a function of $\nabla_\varepsilon g$ and $\nabla_\varepsilon h$, or a combination of all of these terms, depending, as one would suspect, on where in $P(\varepsilon)$ the parameter appears.

Danskin [6,7] provided a now well-known characterization of the directional derivative of the optimal value function of $P(\varepsilon)$ in the case that the constraints are independent of a parameter. Under the conditions that the region of feasibility, R(0), is compact, and $f(x,\varepsilon)$ and $\nabla_\varepsilon f(x,\varepsilon)$ are continuous at $\varepsilon = 0$, Danskin showed that

$$D_z f^*(0) = \min_{x \varepsilon S(0)} \nabla_\varepsilon f(x,0)z. \tag{6.1}$$

Relating our hypotheses to Danskin's construct, we first note the equivalence of our assumption of the uniform compactness of $R(\varepsilon)$ for ε near $\varepsilon = 0$, and the assumption that the feasible region is compact if the constraints of $P(\varepsilon)$ do not depend on ε. To see this, one need only consider that, in this case, $R(\varepsilon) \equiv R(0)$ for all ε and apply Definition 2.3. In addition, when the feasible region is independent of ε, our development need not consider the perturbed point $x^* + \beta(\bar{y} + \delta\tilde{y})$, but may be restricted to the point $x^* \varepsilon S(0)$. The proofs of Corollaries 4.2 and 4.7 remain valid in this case by simply suppressing all reference to the dependence of the constraints on ε and by considering the unperturbed point x^* instead of $x^* + \beta(\bar{y} + \delta\tilde{y})$. One is then led to conclude that, analogous to (4.6) and (4.7),

$$\limsup_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \leq \min_{x \varepsilon S(0)} \nabla_\varepsilon f(x,0) z, \quad \text{and} \tag{6.2}$$

$$\liminf_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \geq \min_{x \varepsilon S(0)} \nabla_\varepsilon f(x,0) z, \tag{6.3}$$

for any unit vector $z \varepsilon E^k$. Thus it follows that, under the stated conditions, namely the compactness of R and the continuity of $f(x,\varepsilon)$ and $\nabla_\varepsilon f(x,\varepsilon)$ at $\varepsilon = 0$, our results are consistent with those of Danskin in that they verify the existence of $D_z f^*(0)$ and show (from (6.2) and (6.3)) that it can be expressed as in (6.1).

Gauvin and Tolle [13] showed, for programs with right-hand side perturbations, i.e., for programs of the form

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } g_i(x) \geq \varepsilon_i \quad (i=1,\ldots,m), \qquad\qquad P'(\varepsilon) \\ &\qquad h_j(x) = \varepsilon_{m+j} \quad (j=1,\ldots,p), \end{aligned}$$

that although the directional derivative of f^* may not exist, its only consider that, in this case, $R(\varepsilon) \equiv R(0)$ for all ε and apply Definition 2.3. In addition, when the feasible region is independent of ε, our development need not consider the perturbed point $x^* + \beta(\bar{y} + \delta\tilde{y})$, but may be restricted to the point $x^* \varepsilon S(0)$. The proofs of Corollaries 4.2 and 4.7 remain valid in this case by simply suppressing all reference to the dependence of the constraints on ε and by considering the unperturbed point x^* instead of $x^* + \beta(\bar{y} + \delta\tilde{y})$. One is then led to conclude that, analogous to (4.6) and (4.7),

$$\limsup_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \leq \min_{x \varepsilon S(0)} \nabla_\varepsilon f(x,0)z, \quad \text{and} \tag{6.2}$$

$$\liminf_{\beta \to 0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \geq \min_{x \varepsilon S(0)} \nabla_\varepsilon f(x,0)z, \tag{6.3}$$

for any unit vector $z \varepsilon E^k$. Thus it follows that, under the stated conditions, namely the compactness of R and the continuity of $f(x,\varepsilon)$ and $\nabla_\varepsilon f(x,\varepsilon)$ at $\varepsilon = 0$, our results are consistent with those of Danskin in that they verify the existence of $D_z f^*(0)$ and show (from (6.2) and (6.3)) that it can be expressed as in (6.1).

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that although the directional derivative of f^* may not exist, its limit quotient can be bounded. In particular, they concluded that if MFCQ holds at each element of S(0) and if R(ε) is uniformly compact for ε near ε = 0, the following inequalities are satisfied:

$$\limsup_{\beta\to0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \le \inf_{x\varepsilon S(0)} \max_{(\mu,\omega)\varepsilon K(x,0)} \left(\sum_{i=1}^{m} \mu_i z_i - \sum_{j=1}^{p} \omega_j z_{m+j} \right), \quad (6.4)$$

and

$$\liminf_{\beta\to0^+} \frac{f^*(\beta z) - f^*(0)}{\beta} \ge \inf_{x\varepsilon S(0)} \min_{(\mu,\omega)\varepsilon K(x,0)} \left(\sum_{i=1}^{m} \mu_i z_i - \sum_{j=1}^{p} \omega_j z_{m+j} \right). \quad (6.5)$$

Now, from (4.6) and (4.7) we see that the bounds we have given for the general program P(ε) reduce to those in (6.4) and (6.5) respectively for the more restrictive perturbations appearing in P'(ε).

In the case of convex programs, the existence of $D_z f^*(0)$ assured by Theorem 4.12 and its expression as (4.10), corresponds under slightly different assumptions, with results achieved by Gol'stein [14] and Hogan [15]. Theorem 4.12 is a direct extension to the general perturbed mathematical program of a result given by Gauvin and Tolle [13] for right-hand side programs.

A. Auslender [4] has extended the results of Gauvin and Tolle [13] to problems involving non-differentiable functions. In particular, the bounds noted by (6.4) and (6.5) are obtained for right-hand side programs in which the problem functions are locally Lipschitz and those defining the equality constraints are continuously differentiable.

Subsequent to the completion of this paper, it came to our attention that Geraud Fontanie ["Locally Lipschitz Functions and Nondifferentiable Programming," M.S. Thesis, Technical Report 80-3, Curriculum in operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 1980] extended the Gauvin-Tolle bounds [13] to a generally perturbed Lipschitz program, using the reduction technique described here.

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